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ASYMPTOTIC THEORY OF A WAVE PACKET IN A BOUNDARY LAYER ON A PLATE*

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The propagation of a wave packet generated by a point source in a boundary layer on a flat plate is considered. The fluid is assumed to be incompressible, and the distance from the leading edge of the plate is chosen to be so large, that the Reynolds number can be assumed to tend to infinity. The field of perturbed motion is constructed using the framework of the linearized theory of the boundary layer with selfinduced pressure, with help of expansions in Laplace integrals with respect to time and Fourier integrals with respect to two spatial variables. The saddle-point method is used to calculate the inverse transforms.

The pulsating motion of the fluid in the wave packet (laminar vortex spot) is characterized by a continuous frequency spectrum. The other special property of the wave packet is that the oscillations are modulated already in the linear stage of its propagation, and thanks to this the amplitude has a sharp maximum at the centre of the perturbed region. The mutual interaction of the wave with continuous distribution of frequencies and wave lengths means that the spectrum of combinative tones is also continuous. The data from the experiments where several isolated harmonics were superimposed /1/** (**Kazanov Yu.S., Kozlov V.V. and Levchenko V.Ya., Experiments on non-linear wave interaction in a boundary layer. Preprint 16, Novosibirsk, In-t teoret. i prikl. mekhaniki, SO AN SSSR, 1978.) show that when the oscillation amplitude increases and the non-linear stage of the process is reached, it is the amplitude of the combination tones that grows most rapidly. The amplitude of the fundamental harmonics grows more slowly. This leads one to the conclusion that the transition from laminar to turbulent flow of the fluid must occur within the wave packet very violently. Indeed, the measurements in /2/ show that the non-linear amplification of the originally monochromatic Tollmien-Schlichting (TS) wave from the unstable frequency range accompanied by the appearance of turbulent pulsations, lasts much longer than the explosive collapse of the wave packet and its transformation into a turbulent spot.

It is very propable that turbulent spots develop from the wave packets at the end of the non-linear stage of the laminar motion /3, 4/. This assumption is reinforced by the definite resemblance, mentioned in /5/, between the isolated spot in laminar flow and a laminar wave packet investigated in /6/. The pulsations occurring within the spot in the frequency range inherent in the selfexciting TS waves strenghten this resemblance.

1. Equations and boundary conditions. In order to simplify the mathematical analysis of the wave packets, we shall assume that the Reynolds number $R \to \infty$. Then the initial Navier-Stokes equations will be reduced asymptotically to the simpler Prandtl equations, with selfinduced pressure remaining to be determined /7-9/. In connection with the three-

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dimensional non-stationary boundary layer in an incompressible fluid in a flat plate, the named equations state that /10, 11/

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \quad \frac{\partial p}{\partial y}=0 \tag{1.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{\partial p}{\partial z}+\frac{\partial^2 w}{\partial y^2}$$

Here t is the time, x, y, z are the Cartesian spatial coordinates and u, v, w are the components of the velocity vector measured in a special dimensionless system of units. The matching conditions show that at the outer edge of the region in question

$$u-y\to A\,(t,x,z),\ w\to 0 \text{ as } y\to\infty \tag{1.2}$$

and the selfinduced pressure p is connected with the displacement thickness A by the relation

$$p\,(t,x,z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}\frac{\partial^2 A\,(t,\xi,\zeta)/\partial\xi^2}{[(x-\xi)^2+(z-\zeta)^2]^{1/2}}\,d\xi \tag{1.3}$$

We will assume that the perturbations are introduced into the boundary layer by means of a localized impulse applied through a small hole in the plate. In this case

$$u=w=0, \quad v=\delta v_0\,(t,x,z) \text{ when } y=0 \tag{1.4}$$

and the function v_0 will be different from zero only for small $t>0$ inside the circle $r=\sqrt{x^2+z^2}<r_0$. Upstream from the source

$$u\to y, \quad p\to 0 \text{ as } x\to-\infty \tag{1.5}$$

The boundary value problem formulated here models the experimental conditions very accurately, except for the fact that the measurements are carried out at moderate values of the Reynolds number /6/*.(*Gilev V.M., Kaganov Yu.S. and Kozlov V.V., Development of a three-dimensional wave packet in a boundary layer. Preprint 34. Novosibirsk, In-t teoret. i prikl. mekhaniki, SO AN SSSR, 1981). This means that only a qualitative, and not a quantitative comparison can be made between the theoretical results and experimental data.

To obtain better agreement, we must turn to the system of linearized Navier-Stokes equations. This was attempted in /12/, but the reasoning behind the assumptions on which the paper was based remains open to doubt. The difficulties were analysed in /13/ where a simpler structure of a two-dimensional wave packet was discussed, which was generated by switching on a vibrator in the form of an infinite strip stretched along the z axis on the plate.

Following /13/ and taking into account (1.2) and (1.5), we shall write

$$(p,A,u-y,v,w)=\delta\,(p',A',u',v',w') \tag{1.6}$$

and linearize the equations of motion of the fluid over the source amplitude. Eliminating the displacement thickness from (1.2) and (1.3) we obtain, as $y\to\infty$,

$$\frac{\partial^2 u'}{\partial x^2}\to\frac{1}{2\pi}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial z^2}\right)\int_{-\infty}^{\infty}d\zeta\int_{-\infty}^{\infty}\frac{p'\,(t,\xi,\zeta)}{[(x-\xi)^2+(z-\zeta)^2]^{1/2}}\,d\xi \tag{1.7}$$

2. Integral transformations.

Let us expand the new functions sought, introduced by means of (1.6), in Laplace integrals in time, and Fourier integrals in the spatial coordinates lying in the plane of the plate. We have

$$[\bar p\,(\omega,k,m),\bar u\,(y;\omega,k,m),\bar v\,(y;\omega,k,m),\bar w\,(y;\omega,k,m)]=$$
$$\int_0^{\infty}e^{-\omega t}\,dt\int_{-\infty}^{\infty}dz\int_{-\infty}^{\infty}dx\,e^{-i(kx+mz)}\times$$
$$[p'\,(t,x,z),u'\,(t,x,y,z),v'\,(t,x,y,z),w'\,(t,x,y,z)]$$

Substituting the above formulas into the linearized Eqs.(1.1) and taking into account the boundary conditions (1.4), (1.5) and (1.7), we obtain for the function transforms $\bar p, \bar u, \bar v$ and $\bar w$ a system of ordinary differential equations. Integration of the system follows the scheme given in /14/ based on the limiting form of the Squire's transformation as $R\to\infty$.

With this purpose in mind we shall determine, for real k and m, the reduced frequency and the wave number by means of the formulas

$$\omega' = (k'/k)^{2/3}\omega, \qquad k' = \operatorname{sign} k\, |k|^{1/4} (k^2 + m^2)^{3/8} \tag{2.1}$$

which can be used to obtain an expression for excess pressure in the form

$$p' = -\frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty} dm \int\limits_{-\infty}^{\infty} dk e^{i(kx+mz)} \times \frac{1}{2\pi i}\int\limits_{b-i\infty}^{b+i\infty} \frac{(ik)^{4/3}\bar{v}_0(\omega, k, m)}{(k^2+m^2)^{1/2} F(\omega', k')} e^{\omega t} d\omega \tag{2.2}$$

Here $\bar{v}_0$ is the transform of the source from (1.4), the quantity F is determined from the equations

$$F = \Phi(\Omega) - Q(k'), \quad \Omega = \omega'(ik')^{-2/3} = \omega(ik)^{-2/3} \tag{2.3}$$

$$\Phi = \frac{d\,\mathrm{Ai}(\Omega)}{dz} I^{-1}(\Omega), \qquad I = \int\limits_{\Omega}^{\infty} \mathrm{Ai}(z)\, dz, \qquad Q = (ik')^{1/3}|k'|$$

and $\mathrm{Ai}(z)$ is the Airy function which tends exponentially to infinity in the sector $-\pi/3 < \arg z < \pi/3$.

Equating to zero the denominator in the integrand in (2.2) we obtain the dispersion relation

$$\Phi(\Omega) = Q(k') \tag{2.4}$$

connecting the complex frequency ω with the wave numbers k and m of the natural spatial oscillations of the boundary layer. Exactly the same relation is obtained in the limit as $R \to \infty$ for the frequency ω' and wave number k' of the two-dimensional TS waves /13/. In the latter case, relation (2.4) has an enumerable number of roots $\omega_n'(k') = (ik')^{2/3}\Omega_n(k')$. Only the first of these roots generates the unstable perturbations, since when $|k'| > k_*' = 1{,}0005$, we have the inequality $\operatorname{Re}\omega_1''(k') > 0$.

Using definitions (2.1) for ω' and k' and (2.3) for the invariant Ω, we conclude that out of the whole collection of spatial oscillation modes, we can have the case when only the first of $\omega_1(k, m) = (ik)^{2/3}\Omega_1(k')$ will be unstable. Furthermore, when k and m are real, the inequality $|\operatorname{Re}\omega_1(k, m)| \leqslant |\operatorname{Re}\omega_1'(k')|$ will hold by virtue of the inequality $|k| \leqslant |k'|$. Calculations carried out in /15/ have shown that positive maxima of the function $\operatorname{Re}\omega_1'(k')$ exist at the points $|k'| = k_2'^* = 2{,}716$ and $|k'| = k_4'^* = 4{,}346$, and the first of two pairs of points shown corresponds to the two-dimensional free TS waves with the largest amplitude increment over time. The spatial oscillations growth increments with $m \neq 0$ are smaller than $\operatorname{Re}\omega_1'(k_2'^*)$.

3. Analysis of the inverse transforms. The first stage in evaluating the integrals in (2.2) consists in finding an appropriate expression for the inverse Laplace transform. Just as was done in /15/ in the problem of the development of two-dimensional perturbations in a boundary layer, we can estimate the contribution to the solution of the sum of residues dependent on all roots of the dispersion relation beginning with the second root. The sum is of order $o(t^{-5})$ and the estimate is uniform in x and z. This leads us to the conclusion that when the time is sufficiently long, the sum becomes insignificant as compared with the contribution of the residue associated with the first root $\omega_1(k, m)$, since the amplitude of the wave packet described by it increases exponentially /13/. As a result, we have

$$p'(t, x, z) = \operatorname{Re}[p_c'(t, x, z)] \tag{3.1}$$

$$p_c' = \frac{1}{2\pi^2}\int\limits_{-\infty}^{\infty} dm \int\limits_{0}^{\infty} dk \exp[\omega_1(k, m)t + ikx + imz] \times \frac{k^2\bar{v}_0[\omega_1(k, m), k, m]}{\sqrt{k^2+m^2}\, d\Phi[\Omega_1(k')]/d\Omega}$$

In order to simplify the subsequent calculations we shall carry out /16/ the following operations on the variables of integration: change to polar coordinates $k = \rho\cos\alpha$, $m = \rho\sin\alpha$, $-\pi/2 \leqslant \alpha \leqslant \pi/2$, stretching of the radius vector $\rho = s\cos^{-1/4}\alpha$, and trigonometric transformation $\alpha = \operatorname{arctg}\beta$, $-\infty < \beta < \infty$. In the new variables we have

$$k' = s, \qquad \omega_1(k, m) = \omega_1'(s)(1 + \beta^2)^{-1/4}$$

and the integral from (3.1) will become

$$p_c' = \frac{1}{2\pi^2}\int\limits_{-\infty}^{\infty} J(\beta, T, X) \frac{d\beta}{(1+\beta^2)^{13/8}} \tag{3.2}$$

$$J = \int_0^\infty \frac{s^2\bar{v}_0[\omega_1'(s)(1+\beta^2)^{-1/4}, s(1+\beta^2)^{-3/4}, s\beta(1+\beta^2)^{-3/4}]}{d\Phi[\Omega_1(s)]/d\Omega} \times$$
$$\exp[\omega_1'(s)T + isX]\,ds; \quad T = (1+\beta^2)^{-1/4}t,$$
$$X = (1+\beta^2)^{-3/4} \quad (x+\beta z)$$

Here the inner integral is of the same type as that appearing in the linear problem of the development of two-dimensional perturbations.

Thus the solution of the auxiliary problem of two-dimensional perturbations formulated in terms of reduced variables T and X, occupies a central position in the process of constructing a structure of the three-dimensional wave packet. A detailed analysis of this problem for moderate values of T and any $V = X/T$, was given in /13/ where a special example based on the ideas of the saddle-point method was used to calculate J.

Let $\varphi(s; V) = \omega_1'(s) + isV$. When V is arbitrary and fixed, the coordinates of the saddle points in the complex s plane satisfy the equation $d\varphi/ds = d\omega_1'/ds + iV = 0$ whose solution $s = S(V)$ splits into an infinite number of branches. Computations have shown that when $T \geqslant 3$, the large amplitude perturbations are concentrated in the central region $3{,}5 \leqslant V \leqslant 7{,}0$. For such values of reduced time J can be calculated using the saddle-point method and the asymptotic form of this integral will be determined by the first branch $S_1(V)$ of the saddle points. We have /13, 17/

$$J = \left(\frac{2\pi}{T}\right)^{1/2} S_1^2 \left[\left|\frac{d^2\varphi(S_1; V)}{ds^2}\right|\right]^{-1/2} \exp[\varphi(S_1; V)T + i\Gamma_{s1}] \times \tag{3.3}$$
$$\frac{\bar{v}_0[\omega_1'(S_1)(1+\beta^2)^{-1/4}, S_1(1+\beta^2)^{-3/4}, S_1\beta(1+\beta^2)^{-3/4}]}{d\Phi[\Omega_1(S_1)]/d\Omega}$$

where $\Gamma_{s1} = \Gamma_{s1}(V)$ is the angle between the positive direction of the s axis and the tangent to the line $\operatorname{Im}\varphi = \text{const}$ drawn through the point S_1. The error in the value of J found using formula (3.3) does not, as a rule, exceed several percent /17/. The asymptotic form obtained is not uniform in β. Indeed, when $|\beta| \to \infty$ and t is fixed, it loses its strength, since according to the third formula of (3.2) the reduced time $T \to 0$.

In order to overcome this difficulty, we shall write p' in the form

$$p_c' = \frac{1}{2\pi^2}\left(\int_{-\infty}^{-\beta_\infty} + \int_{-\beta_\infty}^{\beta_\infty} + \int_{\beta_\infty}^{\infty}\right) J(\beta, T, X)\frac{d\beta}{(1+\beta^2)^{13/4}} \tag{3.4}$$

The computations show that for sufficiently large values of t it is easy to choose a value of the constant β_∞ which will ensure that the contributions from the first and third integral on the right-hand side of (3.4) to p' are small. As regards the second integral, the saddle-point method can again be used to obtain its estimate. The confirmation of the necessary conditions for $z = 0$ is trivial.

Indeed, let

$$\psi(\beta; V_x, V_z) = (1+\beta^2)^{-1/4}\varphi(S_1(V); V), \quad V = (1+\beta^2)^{-1/2}(V_x + \beta V_z)$$
$$V_x = x/t, \quad V_z = z/t$$

Then in the present case the saddle point $\beta = 0$ determined by the equation $d\psi/d\beta = 0$ will lie on the segment of the axis $\operatorname{Im}\beta = 0$ which is the initial contour of integration in the complex β plane. In addition, we can confirm that the maximum value of $\operatorname{Re}\psi$ on the whole contour of integration is reached at precisely the point $\beta = 0$.

The conditions of applicability of the method become less obvious when $z \neq 0$. However, an approximate expression for the second integral from the right-hand side of (3.4) can be obtained without resorting to an asymptotic analysis, since its values can be obtained numerically using any standard method, with a prescribed degree of accuracy relative to the known value of J. The calculations whose results are given below, were carried out within the framework of this approach.

4. Application of the saddle-point method to the double integral. Although the pattern of oscillations of the fluid in a two-dimensional vortex spot can be reliably established using formula (3.3), it is nevertheless useful to derive a similar formula for a spatially localized wave packet. In view of the difficulties mentioned above, we shall return to the initial expression in (3.1), regarding it as a double integral instead of going into the analysis of the complex β plane.

Let us introduce the function

$$\chi(k, m; V_x, V_z) = \omega_1(k, m) + i(kV_x + mV_z) \tag{4.1}$$
$$V_x = x/t \text{ and } V_z = z/t$$

The coordinates of the saddle point in the complex k and m planes satisfy, for any fixed V_x and V_z, the following system of equations:

$$\frac{\partial\chi}{\partial k} = \frac{\partial\omega_1}{\partial k} + iV_x = 0, \quad \frac{\partial\chi}{\partial m} = \frac{\partial\omega_1}{\partial m} + iV_z = 0 \tag{4.2}$$

which admits of a denumerable set of solutions $k = K(V_x, V_z)$, $m = M(V_x, V_z)$. When $V_z = 0$, we have $M(V_x, 0) = 0$. Also $\beta = 0$, and therefore $k = s$, $\omega_1(k, 0) = \omega_1'(s)$ and $V_x = V$. Assuming that the functions in question are continuously dependent on V_z we conclude, using the form of (4.1), that in a sufficiently small neighbourhood of $V_z = 0$ every solution of system (4.2) approaches, infinitely closely, one of the branches of the solution of the equation $d\varphi/ds = 0$. In other words, for the first branch $K_1(V_x, V_z) \to S_1(V_x)$, $M_1(V_x, V_z) \to 0$ as $V_z \to 0$.

Taking into account the definitions (2.1) and (2.3), we shall rewrite the system of Eqs.(4.2) as follows:

$$(ik)^{2/3}\left[\frac{2}{3}\frac{\Omega_1}{k} + \frac{dQ/dk'}{d\Phi(\Omega_1)/d\Omega}\frac{\partial k'}{\partial k}\right] + iV_x = 0 \tag{4.3}$$
$$(ik)^{2/3}\frac{dQ/dk'}{d\Phi(\Omega_1)/d\Omega}\frac{\partial k'}{\partial m} + iV_z = 0$$

and add to it the dispersion relation (2.4). This yields a system of three equations for determining the quantities k, m and Ω_1, depending on the parameters V_x and V_z. Here the existence of the passage to the limit as $V_z \to 0$, plays a central part in organizing the iterative process. The saddle points $K_1(V_x, V_z + \Delta V_z)$ and $M_1(V_x, V_z + \Delta V_z)$ are obtained in the process together with the invariant $\Omega_1^s(V_x, V_z + \Delta V_z)$, by applying Newton's method to the previous approximations $K_1(V_x, V_z)$, $M_1(V_x, V_z)$ and $\Omega_1^s(V_x, V_z)$, and the first step in calculating $K_1(V_x, \Delta V_z)$, $M_1(V_x, \Delta V_z)$ and $\Omega_1^s(V_x, \Delta V_z)$ is based on the known solution $K_1(V_x, 0) = S_1(V_x)$, $M_1(V_x, 0) = 0$ and $\Omega_1^s(V_x, 0) = \Omega_1(S_1)$. The choice $\Delta V_z = 0{,}1$ ensures a rapid convergence of the process. After 2-4 iterations, the unknowns are obtained to 4-6 significant figures. The solid and dashed lines in Fig.1 show the results of calculating the level curves of the functions $\operatorname{Re}\chi$ (with the step of 0.1) and $\operatorname{Im}\chi$ (with the step $\pi/2$).

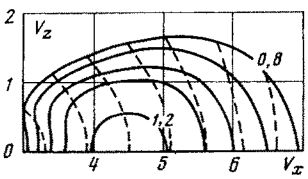

Fig.1

Having established the distribution of the saddle points, we must consider the problem of the presence of singularities in the complex k and m planes. A full investigation would present a very complex problem even for two-dimensional perturbations /13, 17/. It is precisely for this reason that the analysis of the complex β plane was not carried out. However, the rapid convergence of the iterative process which was used to find the roots of the system of Eqs.(4.3) with dispersion Eq.(2.4), implies that when V_z is not too large, no additional singularities will appear in the complex m plane (or they will be at a considerable distance from the saddle points). From this there follows the possibility of replacing, for any fixed V_x and V_z, the initial manifold over which the integration in (3.1) was carried out, with a manifold including the saddle point K_1, M_1. If the maximum value of $\operatorname{Re}\chi$ on the deformed manifold is reached at precisely the point K_1, M_1, then the asymptotic form (3.1) will be expressed for sufficiently large t and finite V_x, V_z, in terms of the contribution from its integration along its closest neighbourhood. In the case of two-dimensional motions the last condition holds within the range $3{,}5 \leqslant V \leqslant 7{,}0$, corresponding to the central part of the perturbed region /17/, but in the case of a spatially localized wave packet its confirmation meets with great difficulties.

Oet us assume that the necessary condition stating that the function $\operatorname{Re}\chi$ reaches its maximum value on the deformed manifold, is satisfied at the point K_1, M_1. Then we can write an appropriate representation for the excess pressure. If we write

$$\Delta = \begin{vmatrix} \dfrac{\partial^2\chi(K_1, M_1; V_x, V_z)}{\partial k^2} & \dfrac{\partial^2\chi(K_1, M_1; V_x, V_z)}{\partial k\partial m} \\ \dfrac{\partial^2\chi(K_1, M_1; V_x, V_z)}{\partial m\partial k} & \dfrac{\partial^2\chi(K_1, M_1; V_x, V_z)}{\partial m^2} \end{vmatrix}$$

then, according to the general theory /18/ we have

$$p_c' = \frac{1}{\pi t}\Delta^{-1/2}\frac{K_1^2 \tilde{v}_0[\omega_1(K_1, M_1), K_1, M_1]}{\sqrt{K_1^2 + M_1^2}\, d\Phi[\Omega_1^s(V_x, V_z)]/d\Omega} \times \exp[\chi(K_1, M_1; V_x, V_z)t] \tag{4.4}$$

Here $K_1 = K_1(V_x, V_z)$ and $M_1 = M_1(V_x, V_z)$ are assumed to have been found using the iterative process described above, and in fact

$$\Omega_1^s(V_x, V_z) = \Omega_1[K_1'(K_1, M_1)],\ K_1' = K_1[1 + (M_1/K_1)^2]^{1/2}$$

The selection of the necessary branches of the multivalued functions K_1' and $\Delta^{1/2}$ is most simply carried out according to the values of their arguments when $V_z = M_1 = 0$, in which case $K_1' = K_1$, $V_x = V$, and the asymptotic form of the integral (3.2) with the expression (3.3) for J together yield

$$\Delta^{1/2} = \left[\left|\frac{\partial^2\chi(K_1, 0; V_x, 0)}{\partial k^2}\right|\left|\frac{\partial^2\chi(K_1, 0; V_x, 0)}{\partial m^2}\right|\right]^{1/2} \times \exp[i(\Gamma_1 + \Gamma_{s1} + \Gamma_{\beta1})]$$

and $\Gamma_1 = \Gamma_1(V_x)$ denotes the argument of $S_1(V_x)$, while $\Gamma_{\beta1} = \Gamma_{\beta1}(V_x)$ is the angle between the direction of the abscissa and the tangent to the line $\operatorname{Im}\psi = \text{const}$ drawn through the origin of coordinates in the complex β plane. The extension of these branches into the complex k and m planes is carried out together with the determination of the coordinates of the saddle points and the invariant Ω_1.

5. Results of computations. For large values of t, influence on the behaviour of the solution $p'(t, x, z)$ is mainly exerted by the exponential factor on the right-hand side of (4.4), while the inverse transform $\bar{v}_0[\omega_1(K_1, M_1), K_1, M_1]$ plays a much smaller part in the formation of its structure. The first of the named quantities represents a universal characteristic of a boundary layer on a plate, and changes with the form of the source of perturbations.

To confirm the assertion made, we calculated the central region of the wave packets generated either by an instantaneous point source, or by sources spread over space and time. The jet of fluid blown into the boundary layer was specified, respectively, either as $v_0 = \delta(t)\delta(x)\delta(z)$ (δ (is the Dirac delta function), or as $v_0 = t^2e^{-t}\cos(\pi x/2)\cos(\pi z/2)$, $-1 \leqslant x \leqslant 1$, $-1 \leqslant z \leqslant 1$. The results for both types of sources show no qualitative differences (in the limit as $t \to \infty$, the statement becomes trivial). Thus it is sufficient to quote only the basic characteristics of the oscillations in the fluid caused by the instantaneous action of an infinitely thin jet.

Both methods described above were used to control the computational data. The pressure calculated according to formula (3.2) where the value of J was obtained using (3.3), was compared with that given by relation (4.4). It was found that the differences were of the order of several percent, provided that $t \geqslant 5$, i.e. the saddle-point method applied to wave packets localized in space, gave approximately the same accuracy as in the case of two-dimensional perturbations /17/.

Fig.2 shows in solid lines the amplitude isolines $|p_c'(t, x, z)|$ of the envelope of the excess pressure peaks (with the step of 0.2), normalized to the quantity $p_0' = 45{,}55$. The dashed lines represent the geometrical sites of the points where $p' = 0$ (every second curve is shown). In this example $t = 5$ and $v_0 = \delta(t)\delta(x)\delta(z)$ (δ (is the Dirac delta function).

The strong similarity between the solid and dashed lines of Fig.1 and 2 leads us to the conclusion that the behaviour of the solution $p'(t, x, z)$ is substantially affected only by $\exp(\chi t)$, while the weight multiplier $K_1^2(K_1^2 + M_1^2)^{-1/2}\Delta^{-1/2}d\Phi[\Omega_1^s]/d\Omega$ and the image $\bar{v}_0[\omega_1(K_1, M_1), K_1, M_1]$ of the source do not materially affect the properties of the oscillations.

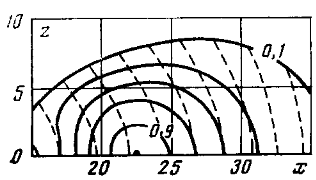

Fig.2

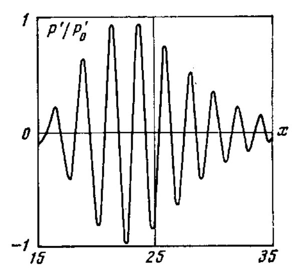

Fig.3

Indeed, the distribution of the amplitude $|p_c'(t, x, z)|$ depends almost completely on $\exp[\operatorname{Re}\chi t]$, at the same time as the form of the wave fronts with $p' = 0$ is determined by $\exp[i\operatorname{Im}\chi t]$. Near the centre of the wave packet the amplitude isolines tend asymptotically to the ellipses and take a more complex form in accordance with the general theory /19/, but in the direction towards the periphery.

The distribution of the oscillations along the central line $z = 0$ (Fig.3 is qualitatively

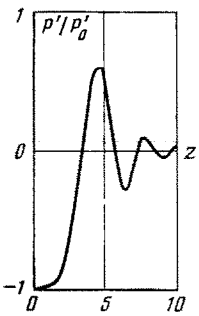

Fig.4

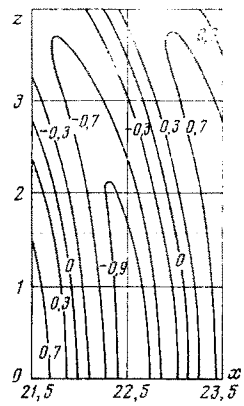

Fig.5

the same as in two-dimensional perturbations /13, 17/. This is explained by the fact that $\chi(K_1, 0; V_x, 0) = \varphi(S_1(V); 0)$, and $K_1 = S_1$, $V_x = V$. On moving away from the central line the fronts fold back, but Fig.2 shows that the foldback is small compared with the corresponding experimental data available from /6, 20/. This implies that the number of oscillations in the transverse direction should be considerably smaller than that in the longitudinal direction. Direct calculations (Fig.4 where $X = 22{,}5$) show that the number is equal to 2-3 only. Fig.5 (the isolines p'/p_0'), offer a more detailed insight into the character of the wave fronts, and also shows good qualitative agreement with the results of the measurements given in /20/.

We find, that within the framework of the proposed asymptotic analysis of a freely self-interacting boundary layer where the Reynolds number $R \to \infty$, the largest amplitude of the oscillations within the wave packet is attained on the central line $z = 0$, and the perturbations do not become forked as long as the linear stage of their propagation persists. At sufficiently large distances from the source the size of the perturbed region increases with time. A half-angle can be shown in the apex of the sector, within which the wave packet propagates. Assuming that the amplitude of the oscillations at the periphery of perturbed region is 10% of the maximum amplitude at its centre, we arrive at the approximate value of 15° for the angle in question.

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THE PRESSURE ON A SPHERE WITH A DAMPING COATING WHEN A PLANE ACOUSTIC WAVE IS INCIDENT ON IT*

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In the problem on the interaction of an acoustic wave with a rigid sphere coated with a thin compressible layer /1/, the non-stationary pressure distribution on the sphere is found. The method of numerical inversion of the Laplace integral transform is used, together with asymptotic relations that hold in the case of a sufficiently thin coating. It is shown that the behaviour of the pressure is qualitatively different in the cases of a rigid sphere and a sphere with a coating. In the pressure-time dependence, successive series of oscillations are discovered, which are not seen with a rigid sphere, see /2, 3/. The pressure rise corresponding to the instant of interaction of the enveloping wave (the "Poisson spot" /4/) is displaced in time and in some cases exceeds twice the incident wave amplitude.

1. Formulation of the problem. Laplace transform of the pressure.

At the instant $t = 0$ let a plane acoustic wave of pressure p_i, previously propagating through a homogeneous fluid at rest with initial pressure p_0, density ρ_0, and sound velocity c_0, be incident on a rigid fixed sphere of radius a, coated with a thin damping layer of initial thickness h_0, where $h_0 \ll a$. The origin of a system of spherical coordinates r, ψ, φ is at the centre of the sphere, the incident wave front is perpendicular to the z axis z ($z = r\cos\varphi$), and the motion is in the negative direction of the axis. For simplicity, the incident wave is regarded as a step with a pressure drop p_m

$$p_i = p_m \eta\left(t + \frac{z-a}{c_0}\right) + p_0, \quad \eta(t) = \begin{cases} 0, t \leqslant 0 \\ 1, t > 0 \end{cases} \tag{1.1}$$

We introduce the dimensionless pressure disturbances $\bar{p}$ and $\bar{p}_i$, the time $\bar{t}$, and coordinate $\bar{r}$ in accordance with the relations

$$\bar{p} = \frac{p - p_0}{p_m}, \quad \bar{p}_i = \frac{p_i - p_0}{p_m}, \quad \bar{t} = \frac{c_0 t}{a}, \quad \bar{r} = \frac{r}{a}$$

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